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THE DIFFUSION OF VORTICITY

FROM A PLANE BOUNDARY

by

A. V. Hershey

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This report is a contribution to numerical ship hydrodynamics.

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THE DIFFUSION OF VORTICITY
FROM A PLANE BOUNDARY

By

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ABSTRACT

Any discontinuity in flow at a plane boundary is dispersed by the diffusion of vorticity into a boundary layer and wake. Experimental measurements in the past have been correlated with empirical formulations. A generalization gives both the horizontal component and the vertical component of the mean velocity. In the laminar sublayer the velocity is a solution of the diffusion equation. In the turbulent boundary layer the velocity can be expressed by a Fourier integral. In the free stream there is a vertical persistence of velocity. The computation of velocity is provided by subroutines.

INTRODUCTION

The drag on a ship is partly a viscous resistance and partly a wave resistance. The viscosity of the fluid causes a transport of momentum from the ship to the fluid. The momentum is located in a boundary layer at the surface of the ship, and in the wake behind the ship. The boundary layer is laminar at the bow, but the boundary layer is turbulent at the stern. For a nonslip boundary condition the velocity relative to the ship is zero at the surface and is the velocity for free flow far from the surface. A knowledge of the velocity may be found from an analysis of the analogous problem in the flow over a plane.

When a thin plate moves edgewise through a viscous fluid the plate entrains fluid if there is a nonslip boundary condition at the surface of the plate. Each differential element of surface of the plate sets up a current which trails downstream and spreads out by diffusion. The velocity at the plate is the accumulated sum of velocities in currents which have been created upstream.

The boundary layer of a plate has been the subject of many investigations¹⁻⁷. There are three principal methods of analysis. The first method is statistical. It leads to a system in which the number of variables is greater than the number of equations. Various schemes for closure have been proposed. The second method is polynomial. The velocity is expressed as a power polynomial in a limited region of a laminar boundary layer. The third method is spectral. The velocity is expressed as a Fourier integral. The rate of change of the Fourier amplitude is determined by an integro-differential equation. The evolution of a velocity distribution from an initial distribution is determined uniquely by the integro-differential equation.

The Fourier analysis has been applied to homogeneous isotropic turbulence. Contact of the fluid with a solid boundary is the origin of real turbulence.

A comprehensive source of information about boundary layers is *Boundary-Layer Theory* by Schlichting⁷. The Fourier analysis of flow in a fluid has been investigated by Batchelor⁸ and by Orszag⁹.

EQUATION OF CONTINUITY

A fluid consists of particles with random velocities. The number of particles per unit volume determines a median density ρ . Across any mathematical boundary there is a flux of particles from both sides. A boundary which is moving at such a speed that the flux is zero defines a median velocity v .

In a continuous fluid the law of conservation of mass requires that within any mathematical boundary the mass density ρ and the particle velocity v are related by the equation

$$\int \frac{\partial \rho}{\partial t} d\tau + \int \rho v \cdot ds = 0 \quad (1)$$

where t is the time, $d\tau$ is a volume element within the boundary, and ds is a surface element on the boundary. Application of the Gauss theorem leads to the equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = \frac{\partial \rho}{\partial t} + v \cdot \nabla \rho + \rho \nabla \cdot v = \frac{d\rho}{dt} + \rho \nabla \cdot v = 0 \quad (2)$$

The divergence of velocity is the rate of expansion of the fluid

EQUATION OF MOTION

The bulk modulus κ is the rate of decrease of pressure per unit rate of expansion. A pressure pulse is propagated through the fluid with the speed of sound $(\kappa/\rho)^{1/2}$.

In a continuous fluid the gradient of velocity satisfies the identity

$$\nabla \mathbf{v} = \frac{1}{2}(\nabla \mathbf{v} - \nabla' \mathbf{v}) + \frac{1}{2}(\nabla \mathbf{v} + \nabla' \mathbf{v}) \quad (3)$$

where $\nabla' \mathbf{v}$ is the transpose of $\nabla \mathbf{v}$. The antisymmetric part is the rate of rotation Ω and the symmetric part is the rate of strain Θ . There would be spinup of fluid particles if stress were not symmetric. In an isotropic fluid the pressure is isotropic and shear stress is proportional to shear rate. The strain rate is partly an isotropic expansion rate, which is the trace of the strain rate, and is partly an anisotropic distortion rate, which is the shear rate. The shear stress arises from the anisotropic distortion rate. The stress Σ is given by the equation

$$\Sigma = -p \mathbf{I} - \frac{2}{3}\mu \nabla \cdot \mathbf{v} \mathbf{I} + 2\mu \Theta \quad (4)$$

where p is the pressure, and μ is the viscosity.

The force on any mathematical boundary is given by the Gauss theorem

$$\int \nabla \cdot \Sigma \, d\tau = \int \Sigma \cdot ds \quad (5)$$

where $d\tau$ is a volume element within the boundary and ds is a surface element at the boundary. The equation of motion per unit volume is given by the equation

$$\rho \frac{d\mathbf{v}}{dt} = \nabla \cdot \Sigma \quad (6)$$

As a consequence of the identity

$$\nabla \cdot \nabla' \mathbf{v} = \nabla \nabla \cdot \mathbf{v} \quad (7)$$

the equation of motion is the Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \int \frac{dp}{\rho} + \frac{1}{3} \frac{\mu}{\rho} \nabla \nabla \cdot \mathbf{v} + \frac{\mu}{\rho} \nabla \cdot \nabla \mathbf{v} \quad (8)$$

The kinematic viscosity ν is given by the equation

$$\nu = \frac{\mu}{\rho} \quad (9)$$

For hydrodynamics the kinematic viscosity is constant. Differentiation throughout the equation of motion leads to the equation

$$\frac{\partial}{\partial t} (\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot \nabla (\nabla \cdot \mathbf{v}) + (\nabla \mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla \cdot \nabla \int \frac{dp}{\rho} + \frac{4}{3} \frac{\mu}{\rho} \nabla \cdot \nabla (\nabla \cdot \mathbf{v}) \quad (10)$$

where a tensor enclosed in parentheses is the contraction of the tensor by internal scalar multiplication. This equation is a Poisson equation for the determination of pressure. Differentiation throughout the equation of motion leads to the equation

$$\frac{\partial}{\partial t} (\nabla \times \mathbf{v}) + \mathbf{v} \cdot \nabla (\nabla \times \mathbf{v}) + [\nabla \mathbf{v} \cdot \nabla \mathbf{v}] = \frac{\mu}{\rho} \nabla \cdot \nabla (\nabla \times \mathbf{v}) \quad (11)$$

where a tensor enclosed in brackets is the contraction of the tensor by internal vector multiplication. This equation is a diffusion equation for the determination of vorticity.

BLASIUS PROFILE

For two-dimensional flow it is fashionable to define Cartesian coordinates x, y, z such that x is parallel to the plane in the direction of flow, y is perpendicular to the plane, and z is parallel to the plane in the direction perpendicular to the flow. Let i, j, k be unit vectors in the directions of increasing x, y, z . The particle velocity v is given by the equation

$$v = ui + vj \quad (12)$$

where the components u, v are zero at the plane but approach $U, 0$ with increasing y . Continuity in an incompressible fluid is expressed by the equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (13)$$

The continuity equation is satisfied when the components of velocity are derived from a stream function ψ in accordance with the equations

$$u = \frac{\partial \psi}{\partial y} \quad v = - \frac{\partial \psi}{\partial x} \quad (14)$$

This follows from the principle that a second-order derivative is independent of the order of differentiation.

For stationary flow the derivative with respect to t is zero. Far downstream derivatives with respect to x approach zero, and the Navier-Stokes equation is reduced to the equation

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad (15)$$

Far downstream the profile of the boundary layer approaches a constant limit.

The limiting profile is given by a relation between two dimensionless variables ξ and ζ , which are defined by the equations

$$\xi = \sqrt{\frac{U}{\nu}} \frac{y}{\sqrt{x}} \quad \psi = \sqrt{\nu U x} \zeta \quad (16)$$

Differentiation leads to the equations

$$u = U \zeta' \quad v = \frac{1}{2} \sqrt{\frac{\nu U}{x}} (\xi \zeta' - \zeta) \quad (17)$$

Then substitution in the reduced equation of motion leads to the Blasius equation

$$\zeta \zeta'' + 2 \zeta''' = 0 \quad (18)$$

Boundary conditions for this differential equation are given by the equations

$$\zeta = \zeta' = 0 \quad \text{at} \quad \xi = 0 \quad (19)$$

$$\zeta' \rightarrow 1 \quad \text{at} \quad \xi \rightarrow \infty \quad (20)$$

In the original derivation by Blasius¹ the variables were so defined that ζ' approached a limit of 2. The derivation herein follows the derivation by Schlichting⁷ where the variables are so defined that ζ' approaches a limit of unity.

An ascending series is given by the equation

$$\zeta = \sum_{m=0}^{\infty} a_m \xi^{3m+2} \quad (21)$$

This series satisfies the boundary conditions at $\xi = 0$. Substitution in the differential equation leads to the recurrence equation

$$\sum_{k=0}^{m-1} (3m - 3k - 2)(3m - 3k - 1)a_k a_{m-k-1} + 2(3m)(3m + 1)(3m + 2)a_m = 0 \quad (22)$$

The recurrence is started with a value of a_0 which is adjusted by trial to make the series meet the boundary conditions at $\xi \rightarrow \infty$. The computed value of a_0 is 0.16603. The ratio between successive coefficients of the series tends to a constant limit with increasing order. It is possible to estimate a remainder after a finite number of terms on the basis of the geometric series. In no case can the series be used for ξ greater than 5.69, where the ratio between terms becomes unity.

For large values of ξ the value of ζ is the sum of a linear term and a correction. Correct to first order the correction is a solution of the equation

$$(\xi + c)\zeta'' + 2\zeta''' = 0 \quad (23)$$

where c is an arbitrary constant. The computed value of c is -1.72077. The solution of the differential equation is given by the equations

$$\zeta = \xi + c - A(\xi + c) \int_{\xi}^{\infty} e^{-\frac{1}{4}(\xi+c)^2} d\xi + 2A e^{-\frac{1}{4}(\xi+c)^2} \quad (24)$$

$$\zeta' = 1 - A \int_{\xi}^{\infty} e^{-\frac{1}{4}(\xi+c)^2} d\xi \quad (25)$$

where A is an arbitrary constant. The arbitrary constant is selected so as to make the corrected linear terms coincide with the series expansion at $\xi = 5.0$. The integral in the correction is given by the equation

$$\int_{\xi}^{\infty} e^{-\frac{1}{4}(\xi+c)^2} d\xi = \sqrt{\pi} [1 - \operatorname{erf}(\frac{1}{2}(\xi + c))] \quad (26)$$

where erf is the error function.

Previous analyses and computations on the Blasius profile are summarized by the data in a table in a paper by Howarth², which is the basis for the Table 7.1 in the text by Schlichting⁷. That table of data is reproduced by the following subroutine.

SUBROUTINE BLSSPF (AU, AN, AX, AY, FU, FV)

 FORTRAN SUBROUTINE FOR BLASIUS PROFILE

The free-stream velocity U is given in argument AU, and the kinematic viscosity ν is given in argument AN. The coordinates x, y are given in the arguments AX, AY. Series expansions and error integrations are used in the evaluation of the Blasius profile. The components u, v are stored in functions FU, FV.

TURBULENT PROFILE

Far downstream the flow in the boundary layer is turbulent. There have been many experiments on the turbulent boundary layer. In the absence of a fundamental theory the experiments have been correlated with empirical formulations. A celebrated formulation is the logarithmic law, which expresses mean velocity as the logarithm of the distance from a wall. The logarithm cannot be used at the wall where velocity is zero and the logarithm is $-\infty$, or at infinite distance from the wall where velocity is finite and the logarithm is $+\infty$. Corrections have been published by Reichardt⁴ and by Thompson⁵. The argument of the logarithm is incremented by unity and the logarithm is blended with the free-stream velocity.

The Reynolds number R_x is defined by the equation

$$R_x = \frac{Ux}{\nu} \quad (27)$$

where U is the free-stream velocity, ν is the kinematic viscosity and x is the distance downstream. The drag D on the wall is given by the equation

$$D = \frac{1}{2} \rho U^2 \frac{cx}{(\log R_x)^{2.58}} \quad (28)$$

where the constant c is defined by the equation

$$c = 0.455 (\log 10)^{2.58} \quad (29)$$

This formulation is equivalent to Equation 21.16 in the text by Schlichting⁷. A shear velocity u_* is defined by the equation

$$u_* = \sqrt{\frac{\tau}{\rho}} \quad (30)$$

where τ is the shear stress at the wall, and ρ is the density. The shear stress τ is given by the equation

$$\tau = \frac{dD}{dx} \quad (31)$$

The boundary layer thickness δ is determined by the equation

$$\frac{u_* \delta}{\nu} = \frac{R_x}{(\log R_x)^{2.58}} \quad (32)$$

This equation is equivalent to Table 21.1 in the text by Schlichting⁷. Finally the mean velocity u is given by the equation

$$u = \frac{\gamma u_*}{\kappa} \log \left(1 + \kappa \frac{u_* y}{\nu} \right) + (1 - \gamma) U \quad (33)$$

where κ is a constant and γ is the blending function. The experimental value of κ is given by the equation

$$\kappa = 0.40 \quad (34)$$

Various empirical schemes for the blending function have been tried, but a more logical basis for the blending function would be the error function. To within

experimental error the blending function can be expressed by the equations

$$\eta = \pi \frac{y}{\delta} \qquad \gamma = \frac{1}{2} \frac{\operatorname{erf}(\eta + \frac{\pi}{2}) - \operatorname{erf}(\eta - \frac{\pi}{2})}{\operatorname{erf}(\frac{\pi}{2})} \quad (35)$$

Then the blending function is in the range $1 > \gamma > 0$ while y is in the range $0 < y < \infty$

Differentiation with respect to x leads to the equation

$$\tau = \frac{1}{2} \rho U^2 \frac{c}{(\log R_x)^{2.58}} \left[1 - \frac{2.58}{\log R_x} \right] \quad (36)$$

and further differentiation leads to the equation

$$\frac{d\tau}{dx} = -\frac{1}{2} \rho U^2 \frac{2.58c}{x(\log R_x)^{3.58}} \left[1 - \frac{3.58}{\log R_x} \right] \quad (37)$$

Then the derivative of u_* is given by the equation

$$\frac{du_*}{dx} = \frac{1}{2} u_* \frac{1}{\tau} \frac{d\tau}{dx} \quad (38)$$

Differentiation with respect to x leads to the equation

$$\frac{d\delta}{dx} = \frac{U}{u_*(\log R_x)^{3.58}} \left[\log R_x - \frac{1}{2}(2.58) \frac{1 - \frac{1.58}{\log R_x}}{1 - \frac{2.58}{\log R_x}} \right] \quad (39)$$

Then the derivative of γ is given by the equation

$$\frac{d\gamma}{dx} = \sqrt{\pi} \frac{y}{\delta} \frac{e^{-(\eta - \frac{\pi}{2})^2} - e^{-(\eta + \frac{\pi}{2})^2}}{\operatorname{erf}(\frac{\pi}{2})} \frac{1}{\delta} \frac{d\delta}{dx} \quad (40)$$

Finally the derivative of u with respect to x is given by the equation

$$\frac{\partial u}{\partial x} = \frac{1}{\kappa} \log \left(1 + \kappa \frac{u_* y}{\nu} \right) \left(u_* \frac{d\gamma}{dx} + \gamma \frac{du_*}{dx} \right) + \frac{\frac{u_* y}{\nu}}{1 + \kappa \frac{u_* y}{\nu}} \gamma \frac{du_*}{dx} - U \frac{d\gamma}{dx} \quad (41)$$

Although the derivatives are given by empirical functions with finite numbers of terms their integration with respect to y would lead to an unlimited number of terms. In a practical integration with respect to y the integrand is best approximated by a power polynomial in y and integrated term by term. A discrete set of integrands is converted into a discrete set of coefficients by 11-point Lagrange interpolation and the integration is completed coefficient by coefficient.

The stream function ψ is given by the equation

$$\psi = \int_0^y u dy \quad (42)$$

and the vertical component v is given by the equation

$$v = - \int_0^y \frac{\partial u}{\partial x} dy \quad (43)$$

The integrations are performed with the aid of the following subroutine

SUBROUTINE TBLNPF (AU, AN, AX, AY, FU, FV)

 FORTRAN SUBROUTINE FOR TURBULENT PROFILE.

The free-stream velocity U is given in argument AU, and the kinematic viscosity ν is given in argument AN. The coordinates x, y are given in the arguments AX, AY. Empirical formulations and Lagrange interpolation are used in the evaluation of the turbulent profile. The components u, v are stored in the functions FU, FV.

LINE FLUX PROFILE

The usual assumption is made that pressure is constant over a flat plate in a steady flow parallel to the plate. Let the free-stream velocity $+U\mathbf{i}$ be disturbed by a small counter velocity $\mathbf{v}$. The Fourier transform can be applied to a function which is zero everywhere except at the origin. Then Fourier integration leads to the equation

$$\mathbf{v} = - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\mathbf{i} + \frac{\lambda}{i\kappa} \mathbf{j} \right) e^{-\lambda x + i\kappa y} d\kappa \quad (44)$$

That this expression for velocity has zero divergence can be verified directly by differentiation. Furthermore it can be derived from the stream function ψ in the equation

$$\psi = - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-\lambda x + i\kappa y}}{i\kappa} d\kappa \quad (45)$$

At $x = 0$ an application of the Euler theorem and an integration of the sine quotient function shows that $\psi = -\frac{1}{2}$ for all $y > 0$ and $\psi = +\frac{1}{2}$ for all $y < 0$. Thus the flux in the counter flow is unity at $x = 0$. The curl of the velocity is given by the equation

$$\nabla \times \mathbf{v} = + \frac{\mathbf{k}}{2\pi} \int_{-\infty}^{+\infty} \frac{(\lambda^2 - \kappa^2)}{i\kappa} e^{-\lambda x + i\kappa y} d\kappa = \gamma \mathbf{k} \quad (46)$$

To within small quantities of second order the vorticity satisfies the differential equation

$$U \frac{\partial \gamma}{\partial x} = \nu \left(\frac{\partial^2 \gamma}{\partial x^2} + \frac{\partial^2 \gamma}{\partial y^2} \right) \quad (47)$$

Substitution of the Fourier integral into the differential equation leads to the equation

$$-U\lambda = \nu (\lambda^2 - \kappa^2) \quad (48)$$

which may be solved by the quadratic rule to give the equation

$$\lambda = - \frac{U}{2\nu} + \sqrt{\left(\frac{U}{2\nu} \right)^2 + \kappa^2} = \frac{2\nu\kappa^2}{U + \sqrt{U^2 + 4\nu^2\kappa^2}} \quad (49)$$

The parameter λ is an even function of the parameter κ . The radical is negative upstream and is positive downstream. Otherwise the integrals do not converge.

When y is increased to infinity the integration with respect to κ makes a significant contribution to the integration only where $\kappa \rightarrow 0$.

Upstream the parameter λ is given by the limit

$$\lambda \rightarrow -\frac{U}{\nu} \quad (\kappa \rightarrow 0) \quad (50)$$

Thus the stream function ψ is given by the limit

$$\psi \rightarrow \mp \frac{1}{2} e^{-\frac{U|x|}{\nu}} \quad (y \rightarrow \pm \infty) \quad (51)$$

and the velocity v is given by the limit

$$v \rightarrow \pm \frac{1}{2} \frac{U}{\nu} e^{-\frac{U|x|}{\nu}} j \quad (y \rightarrow \pm \infty) \quad (52)$$

The vertical component of velocity persists with increasing y , but diminishes rapidly with distance $|x|$ upstream.

Downstream the parameter λ is given by the limit

$$\lambda \rightarrow 0 \quad (\kappa \rightarrow 0) \quad (53)$$

Thus the stream function ψ is given by the limit

$$\psi \rightarrow \mp \frac{1}{2} \quad (y \rightarrow \pm \infty) \quad (54)$$

and the flux in the counter current is everywhere unity. It is independent of distance x downstream.

In the integrals for the components of velocity the integrands are the products of the monotonic factors

$$-\frac{1}{2\pi} e^{-\lambda x} \quad -\frac{\lambda}{2\pi i \kappa} e^{-\lambda x} \quad (55)$$

and the oscillatory factor

$$e^{i\kappa y} \quad (56)$$

The integration through any number of cycles of the oscillatory factor can be completed if the monotonic factors are expressed as power series in κ . The range of the power series is limited by the presence of the radical

$$\sqrt{\left(\frac{U}{2\nu}\right)^2 + \kappa^2} \quad (57)$$

in the parameter λ . For small values of κ the radical is expressed by the equation

$$\sqrt{\left(\frac{U}{2\nu}\right)^2 + \kappa^2} = \frac{U}{2\nu} \sum_{m=0}^{\infty} \frac{(-1)^{m+1} (2m)!}{(2m-1) 2^{2m} (m!)^2} \left(\frac{2\nu\kappa}{U}\right)^{2m} \quad (58)$$

This series converges only when κ meets the limitation

$$-\frac{U}{2\nu} < \kappa < +\frac{U}{2\nu} \quad (59)$$

The series is an even function of κ . For large values of κ the radical is expressed by

the equation

$$\sqrt{\left(\frac{U}{2\nu}\right)^2 + \kappa^2} = \kappa \sum_{m=0}^{\infty} \frac{(-1)^{m+1}(2m)!}{(2m-1)2^{2m}(m!)^2} \left(\frac{U}{2\nu\kappa}\right)^{2m} \quad (60)$$

This series converges only when κ meets the limitation

$$-\frac{2\nu}{U} < \frac{1}{\kappa} < +\frac{2\nu}{U} \quad (61)$$

The series is an odd function of $1/\kappa$. Efficient evaluation of the series is only possible if the ranges of their arguments are much less than the limits of their convergence. The monotonic factors are expanded in each of a sequence of intervals of limited range.

The value of κ in the first interval is given by the equation

$$\kappa = \eta + \theta \quad (62)$$

where η is the center of expansion and θ is the variable of expansion. The variable θ is given by the equation

$$\theta = \epsilon u \quad (-1 \leq u \leq +1) \quad (63)$$

where ϵ is half the range of expansion and u is a variable of interpolation. The first interval straddles the origin where $\eta = 0$, then in subsequent intervals η is incremented by 2ϵ . Thus the monotonic factors are approximated by the series

$$e^{-\lambda x} = \sum_{m=0}^{\infty} a_m u^m \quad \frac{\lambda}{\kappa} e^{-\lambda x} = \sum_{m=0}^{\infty} a_m u^m \quad (64)$$

Required for the computation of velocity are the integrals

$$\epsilon e^{i\eta y} \int_0^{+1} a_m u^m e^{i\epsilon y u} du \quad \epsilon e^{i\eta y} \int_{-1}^{+1} a_m u^m e^{i\epsilon y u} du \quad (65)$$

Required for the integration is the recurrence equation

$$\int_0^{+1} u^m e^{i\epsilon y u} du = \left[\frac{u^m}{i\epsilon y} e^{i\epsilon y u} \right]_0^{+1} - \frac{m}{i\epsilon y} \int_0^{+1} u^{m-1} e^{i\epsilon y u} du \quad (66)$$

The recurrence is started with the initial integral in the equation

$$\int_0^{+1} e^{i\epsilon y u} du = \left[\frac{e^{i\epsilon y u}}{i\epsilon y} \right]_0^{+1} \quad (67)$$

The recurrence is cycled in ascending order if ϵy satisfies the inequality

$$|\epsilon y| \geq 17 \quad (68)$$

Otherwise the recurrence is cycled in descending order with an initial approximation for $m = 64$

The value of κ in the last interval is given by the equation

$$\kappa = -\frac{\delta}{u} \quad (-1 \leq u \leq +1) \quad (69)$$

where δ is the limit of integration. The last interval straddles the point where $\frac{1}{\kappa} = 0$. Continuity through the point is achieved by giving to the radical in λ the same sign

as the sign of κ . The monotonic factors are approximated by the series

$$e^{-(\lambda-\kappa)x} = \sum_{m=0}^{\infty} c_m u^m \quad \frac{\lambda}{\kappa} e^{-(\lambda-\kappa)x} = \sum_{m=0}^{\infty} c_m u^m \quad (70)$$

Then the components of velocity are expressed in terms of the integrals

$$\sum_{m=0}^{\infty} c_m \int_{\delta}^{\infty} \left(-\frac{\delta}{\kappa}\right)^m e^{-\kappa(x-iy)} d\kappa \quad (71)$$

Required for the integration is the recurrence equation

$$\int_{\delta}^{\infty} \frac{1}{\kappa^m} e^{-\kappa(x-iy)} d\kappa = - \left[\frac{e^{-\kappa(x-iy)}}{(m-1)\kappa^{m-1}} \right]_{\delta}^{\infty} - \frac{(x-iy)}{(m-1)} \int_{\delta}^{\infty} \frac{1}{\kappa^{m-1}} e^{-\kappa(x-iy)} d\kappa \quad (72)$$

The recurrence is started with the initial integrals in the equations

$$\int_{\delta}^{\infty} e^{-\kappa(x-iy)} d\kappa = - \left[\frac{e^{-\kappa(x-iy)}}{x-iy} \right]_{\delta}^{\infty} \quad (73)$$

$$\int_{\delta}^{\infty} \frac{1}{\kappa} e^{-\kappa(x-iy)} d\kappa = \int_{-\infty}^{-\delta(x-iy)} \frac{e^t}{t} dt = -\text{Ei}(-\delta(x-iy)) \quad (74)$$

The recurrence is cycled in descending order if $\delta(x-iy)$ satisfies the inequality

$$|\delta(x-iy)| \leq 17 \quad (75)$$

Otherwise the recurrence is cycled in ascending order with an initial approximation for $m = 64$.

For the series expansions the arguments have Chebyshev spacing and interpolations are made with 11-point central Lagrange interpolation. Preliminary computations have established a matrix of coefficients such that the coefficients for progressively increasing powers of the argument are obtained with the product of an array of values of the function and the matrix of coefficients.

If x satisfies the inequality

$$\frac{7U}{8\nu} > \sqrt{\left(\frac{18}{x}\right)^2 + \frac{U}{\nu} \frac{18}{x}} \quad (76)$$

then ϵ and δ are given by the equations

$$\epsilon = \frac{\delta}{7} \quad \delta = \sqrt{\left(\frac{18}{x}\right)^2 + \frac{U}{\nu} \frac{18}{x}} \quad (77)$$

and the last interval can be jettisoned without significant error. Otherwise ϵ and δ are given by the equations

$$\epsilon = \frac{U}{8\nu} \quad \delta = \frac{7U}{8\nu} \quad (78)$$

in which case the Reynolds number is less than 36.

The components of velocity for the line flux are computed by the following subroutine.

SUBROUTINE LNFXPF (AU, AN, AX, AY, FU, FV)

.....
 FORTRAN SUBROUTINE FOR LINE FLUX PROFILE

The free-stream velocity U is given in argument AU, and the kinematic viscosity ν is given in argument AN. The coordinates x, y are given in the arguments AX, AY. Ascending and descending recurrence relations are used in the evaluation of Fourier integrals for a line flux profile. The components u, v of velocity are stored in functions FU, FV.

NONSLIP BOUNDARY

Exploratory computations with LNFXPF have indicated the nature of the flow from a line flux.

If x and y are decreased to zero, the contribution to integration extends to large values of κ where λ approaches the approximation

$$\lambda \rightarrow -\frac{U}{2\nu} \mp |\kappa| \quad (|\kappa| \rightarrow \infty) \quad (79)$$

At the limit of small x and y the velocity is given by the equation

$$v = -\frac{1}{\pi} \frac{|x|\mathbf{i} + y\mathbf{j}}{x^2 + y^2} \quad (x \rightarrow 0, y \rightarrow 0) \quad (80)$$

which is an efflux upstream and an influx downstream

When x is increased to infinity downstream, the integration with respect to κ makes a significant contribution to the integration only where $\kappa \rightarrow 0$ and λ approaches the approximation

$$\lambda \rightarrow \frac{\nu}{U} \kappa^2 \quad (\kappa \rightarrow 0) \quad (81)$$

Then the velocity is given by the equation

$$v = -\sqrt{\frac{U}{4\pi\nu x}} \left(\mathbf{i} + \frac{y}{2x} \mathbf{j} \right) e^{-\frac{Uy^2}{4\nu x}} \quad (x \rightarrow \infty) \quad (82)$$

This is a Gaussian distribution of velocity.

In a continuous distribution of line fluxes the efflux and the influx at the leading edge would tend to cancel. It is assumed that for integrated distributions the Gaussian profile is adequate.

The formula for velocity does not meet a nonslip boundary condition because the velocity at $y = 0$ varies inversely as the square root of distance downstream. However, a continuous distribution of line fluxes may be integrated to give a constant velocity. As a consequence of the equation

$$\int_0^x \frac{d\alpha}{\sqrt{\alpha} \sqrt{x-\alpha}} = 2 \left[\tan^{-1} \sqrt{\frac{\alpha}{x-\alpha}} \right]_0^x = \pi \quad (83)$$

the integrated velocity is given by the equation

$$\sqrt{\frac{4\nu U}{\pi}} \int_0^x v(x-\alpha) \frac{d\alpha}{\sqrt{\alpha}} = -U\mathbf{i} \quad (y = 0) \quad (84)$$

as required by the nonslip boundary condition. Thus the strength of line flux per unit displacement is given downstream by the expression

$$\sqrt{\frac{4\nu U}{\pi\alpha}} \quad (85)$$

The Gaussian distribution is equivalent to the dispersion of a line pulse of vorticity which has diffused outward for a time equal to the expression

$$\frac{x - \alpha}{U} \quad (86)$$

and has been swept downstream with the speed U

FOURIER TRANSFORM

Let $f(x)$ be a periodic function of x . Then it can be approximated by the Fourier series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad (87)$$

Inasmuch as cosines are even and sines are odd, they satisfy the equation

$$\int_{-\pi}^{+\pi} \cos kx \sin mx \, dx = 0 \quad (88)$$

Application of the Euler theorem to the following integrals

$$\frac{1}{2} \int_{-\pi}^{+\pi} e^{i(k+m)x} \, dx \pm \frac{1}{2} \int_{-\pi}^{+\pi} e^{i(k-m)x} \, dx = \frac{1}{2} \left[\frac{e^{i(k+m)x}}{i(k+m)} \right]_{-\pi}^{+\pi} \pm \frac{1}{2} \left[\frac{e^{i(k-m)x}}{i(k-m)} \right]_{-\pi}^{+\pi} = 0 \quad (k \neq m) \quad (89)$$

shows that the trigonometric functions satisfy the orthogonality relations

$$\int_{-\pi}^{+\pi} \cos kx \cos mx \, dx = \int_{-\pi}^{+\pi} \sin kx \sin mx \, dx = 0 \quad (90)$$

Application of the addition theorem leads to the equations

$$\int_{-\pi}^{+\pi} dx = 2\pi \quad (91)$$

$$\int_{-\pi}^{+\pi} \cos^2 kx \, dx = \int_{-\pi}^{+\pi} \sin^2 mx \, dx = \pi \quad (92)$$

The mean square error for the Fourier series is given by the equation

$$\sigma^2 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \left\{ \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx - f(x) \right\}^2 dx \quad (93)$$

Differentiation with respect to the coefficients shows that for least squares error the coefficients are given by the equations

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos nx \, dx \quad b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin nx \, dx \quad (94)$$

Because of the symmetry of the trigonometric functions, the trigonometric series is

given by the equation

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{n=+\infty} \int_{-\pi}^{+\pi} f(s) \cos n(x-s) ds = \frac{1}{2\pi} \sum_{n=-\infty}^{n=+\infty} \int_{-\pi}^{+\pi} f(s) e^{in(x-s)} ds \quad (95)$$

The substitutions

$$x \rightarrow \frac{2\pi x}{L} \qquad s \rightarrow \frac{2\pi s}{L} \quad (96)$$

expand the range of approximation from 2π to L . The functions $f(x)$ and $f(s)$ are replaced by the functions $F(x)$ and $F(s)$ as expressed by the equation

$$F(x) = \frac{1}{L} \sum_{n=-\infty}^{n=+\infty} \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} F(s) e^{i\frac{2\pi n}{L}(x-s)} ds \quad (97)$$

The substitution

$$\kappa = \frac{2\pi n}{L} \quad (98)$$

replaces summation with respect to n by integration with respect to κ in the limit as $L \rightarrow \infty$. The approximation of summation by integration requires the functions to have properties of integrability. The Fourier series is approximated by the Fourier transform

$$A(\kappa) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(x) e^{-i\kappa x} dx \quad (99)$$

$$F(x) = \int_{-\infty}^{+\infty} A(\kappa) e^{i\kappa x} d\kappa \quad (100)$$

In a multidimensional space there is a Fourier transform for each coordinate. Each transform in a series of transforms can be applied to the amplitude of the previous transform.

The two-dimensional transform is given by the equations

$$A(\alpha, \beta) = \frac{1}{4\pi^2} \iint F(x, y) e^{-i(\alpha x + \beta y)} dx dy \quad (101)$$

$$F(x, y) = \iint A(\alpha, \beta) e^{i(\alpha x + \beta y)} d\alpha d\beta \quad (102)$$

Let $\varphi(x, y, z)$ be defined by the equation

$$\varphi(x, y, z) = \iint A(\alpha, \beta) e^{-\sqrt{\alpha^2 + \beta^2}|z| + i(\alpha x + \beta y)} d\alpha d\beta \quad (103)$$

Then φ is a solution of Laplace's equation

$$\nabla \cdot \nabla \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \quad (104)$$

wherever $z \neq 0$. The derivative of φ with respect to z is given by the equation

$$-\frac{\partial \varphi}{\partial z} = \pm \iint \sqrt{\alpha^2 + \beta^2} A(\alpha, \beta) e^{-\sqrt{\alpha^2 + \beta^2}|z| + i(\alpha x + \beta y)} d\alpha d\beta \quad (105)$$

where the sign is $-$ for $z < 0$ and the sign is $+$ for $z > 0$. The difference in the derivative

on opposite sides of the plane $z = 0$ is $4\pi\sigma$ where σ is the source density on the plane. The amplitude

$$2\sqrt{\alpha^2 + \beta^2} A(\alpha, \beta) \quad (106)$$

is the amplitude for $4\pi\sigma$. For a unit source at the origin the amplitude is given by the equation

$$A(\alpha, \beta) = \frac{1}{2\pi\sqrt{\alpha^2 + \beta^2}} \quad (107)$$

Two solutions of Laplace's equation are identical within an additive constant if they have the same normal derivative on a boundary. Thus the inverse of distance is given by the equation

$$\frac{1}{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{2\pi} \iint \frac{1}{\sqrt{\alpha^2 + \beta^2}} e^{-\sqrt{\alpha^2 + \beta^2}|z| + i(\alpha x + \beta y)} d\alpha d\beta \quad (108)$$

Application of the Fourier transform to the real exponential factor in the integrand leads to the equation

$$\frac{1}{r} = \frac{1}{2\pi^2} \iiint \frac{1}{\alpha^2 + \beta^2 + \gamma^2} e^{i(\alpha x + \beta y + \gamma z)} d\alpha d\beta d\gamma \quad (109)$$

The evaluation of this Fourier integral can be completed after a transformation of coordinates.

Let κ, θ, ϕ be polar coordinates with polar axis in the direction of r . Then α, β, γ are given by the equations

$$\alpha = \kappa \sin \theta \cos \phi \quad \beta = \kappa \sin \theta \sin \phi \quad \gamma = \kappa \cos \theta \quad (110)$$

$$d\alpha d\beta d\gamma \rightarrow \kappa^2 \sin \theta d\kappa d\theta d\phi \quad (111)$$

and integration with respect to θ, ϕ leads to the equation

$$\frac{1}{r} = \frac{1}{2\pi^2} \iint \int \left(\frac{1}{\kappa^2} \right) e^{i\kappa r \cos \theta} \kappa^2 \sin \theta d\kappa d\theta d\phi = \frac{2}{\pi} \int_0^\infty \frac{\sin \kappa r}{\kappa r} d\kappa \quad (112)$$

This equation confirms the existence of a Fourier transform for the potential of a pole.

Let r, θ, ϕ be polar coordinates with polar axis in the direction of κ . Then x, y, z are given by the equations

$$x = r \sin \theta \cos \phi \quad y = r \sin \theta \sin \phi \quad z = r \cos \theta \quad (113)$$

$$dx dy dz \rightarrow r^2 \sin \theta dr d\theta d\phi \quad (114)$$

Another solution of Laplace's equation is given by the equation

$$-\frac{\partial}{\partial z} \left(\frac{1}{r} \right) = \frac{\cos \theta}{r^2} \quad (115)$$

Its Fourier amplitude is given by the equation

$$A(\kappa) = \frac{1}{8\pi^3} \iiint \left(\frac{\cos \theta}{r^2} \right) e^{-i\kappa r \cos \theta} r^2 \sin \theta dr d\theta d\phi \quad (116)$$

Integration with respect to θ, ϕ is completed with an integration by parts to give the

equation

$$A(\kappa) = \frac{i}{2\pi^2\kappa} \int_0^\infty \left\{ \cos(\kappa r) - \frac{\sin(\kappa r)}{\kappa r} \right\} \frac{dr}{r} \quad (117)$$

Integration with respect to r is completed with an integration by parts to give the equation

$$A(\kappa) = \frac{i}{2\pi^2\kappa} \left[\frac{\sin(\kappa r)}{\kappa r} \right]_0^\infty = -\frac{i}{2\pi^2\kappa} \quad (118)$$

This equation confirms the existence of a Fourier transform for the potential gradient of a pole.

The gradient of inverse distance is given by the equation

$$\nabla\left(\frac{1}{r}\right) = -\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\{x^2 + y^2 + z^2\}^{\frac{3}{2}}} = \frac{i}{2\pi^2} \int \int \int \frac{\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}}{\alpha^2 + \beta^2 + \gamma^2} e^{i(\alpha x + \beta y + \gamma z)} d\alpha d\beta d\gamma \quad (119)$$

The velocity of a point vortex of unit strength is given by the equation

$$\mathbf{v} = \frac{1}{4\pi} \mathbf{k} \times \nabla\left(\frac{1}{r}\right) = \frac{1}{4\pi} \frac{y\mathbf{i} - x\mathbf{j}}{\{x^2 + y^2 + z^2\}^{\frac{3}{2}}} = -\frac{i}{8\pi^3} \int \int \int \frac{\beta\mathbf{i} - \alpha\mathbf{j}}{\alpha^2 + \beta^2 + \gamma^2} e^{i(\alpha x + \beta y + \gamma z)} d\alpha d\beta d\gamma \quad (120)$$

This equation confirms the existence of a Fourier transform for the velocity of a point vortex.

The circulation around a circle at coordinates r, θ is given by the equation

$$\oint \mathbf{v} \cdot d\mathbf{r} = -\frac{1}{2} \frac{\sin^2 \theta}{r} \quad (121)$$

A change from spherical polar coordinates to cylindrical polar coordinates and integration with respect to z with the aid of the equation

$$\int_{-\infty}^{+\infty} \frac{x^2 + y^2}{z^2} \frac{dz}{\sqrt{x^2 + y^2 + z^2}} = -\left[\frac{\sqrt{x^2 + y^2 + z^2}}{z} \right]_{-\infty}^{+\infty} \quad (122)$$

confirms that the circulation is unity around a line vortex of unit strength and of infinite length.

INCOMPRESSIBLE FLUID

Let $\mathbf{r}$ be a position vector in physical space and let $\boldsymbol{\kappa}$ be a position vector in wave number space. The position vectors are defined by the equations

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad \boldsymbol{\kappa} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k} \quad (123)$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are orthogonal unit vectors.

Let the velocity vector $\mathbf{v}$ at the position vector $\mathbf{r}$ be expressed by the equation

$$\mathbf{v}(\mathbf{r}) = \int A(\boldsymbol{\kappa}) e^{i\boldsymbol{\kappa} \cdot \mathbf{r}} |d\boldsymbol{\kappa}| \quad (124)$$

where $\boldsymbol{\kappa}$ is a vector in wave number space, $|d\boldsymbol{\kappa}|$ is a volume element in wave number space, and $A(\boldsymbol{\kappa})$ is the amplitude of the Fourier component with wave number $\boldsymbol{\kappa}$. The

amplitude is given by the equation

$$A(\kappa) = \frac{1}{8\pi^3} \int v(r) e^{-i\kappa \cdot r} |dr| \quad (125)$$

where $|dr|$ is a volume element in physical space. The gradient of velocity is given by the equation

$$\nabla v = i \int \kappa A(\kappa) e^{i\kappa \cdot r} |d\kappa| \quad (126)$$

the divergence of velocity is given by the equation

$$\nabla \cdot v = i \int \kappa \cdot A(\kappa) e^{i\kappa \cdot r} |d\kappa| \quad (127)$$

and the curl of velocity is given by the equation

$$\nabla \times v = i \int \kappa \times A(\kappa) e^{i\kappa \cdot r} |d\kappa| \quad (128)$$

The Laplacian of velocity is given by the equation

$$\nabla \cdot \nabla v = - \int \kappa^2 A(\kappa) e^{i\kappa \cdot r} |d\kappa| \quad (129)$$

The amplitude $\kappa \cdot A(\kappa)$ of the divergence $\nabla \cdot v$ is given directly by the equation

$$\kappa \cdot A(\kappa) = \frac{1}{8\pi^3} \int \nabla \cdot v e^{-i\kappa \cdot r} |dr| \quad (130)$$

If the divergence is zero for every r in physical space, the amplitude is zero for every κ in wave number space. The amplitude satisfies the orthogonality equation

$$\kappa \cdot A(\kappa) = 0 \quad (131)$$

If a nonzero divergence did happen to occur in an incompressible fluid, a pressure pulse would be created, and the nonzero divergence would be dispersed by a potential flow

Terms which are quadratic in velocity are expressed by double integrals. Let κ_1 and κ_2 be variables of integration in the double integration. Incompressibility is expressed by the equations

$$\kappa_1 \cdot A(\kappa_1) = 0 \quad (\kappa_1 - \kappa_2) \cdot A(\kappa_1 - \kappa_2) = 0 \quad \kappa_2 \cdot A(\kappa_2) = 0 \quad (132)$$

When terms with the same wave number are collected in the integration the amplitudes are collected in a convolution. The product of velocity and its gradient is given by the equation

$$v \cdot \nabla v = i \int \int A(\kappa_2) \kappa_1 \cdot A(\kappa_1 - \kappa_2) e^{i\kappa_1 \cdot r} |d\kappa_1| |d\kappa_2| \quad (133)$$

The square of the gradient is given by the equation

$$(\nabla v \cdot \nabla v) = - \int \int A(\kappa_2) \cdot \kappa_1 \kappa_1 \cdot A(\kappa_1 - \kappa_2) e^{i\kappa_1 \cdot r} |d\kappa_1| |d\kappa_2| \quad (134)$$

This scalar is the divergence of the vector

$$i \int \int A(\kappa_2) \cdot \frac{\kappa_1 \kappa_1 \kappa_1}{\kappa_1 \cdot \kappa_1} \cdot A(\kappa_1 - \kappa_2) e^{i\kappa_1 \cdot r} |d\kappa_1| |d\kappa_2| \quad (135)$$

The gradient of pressure is given by the equation

$$\nabla p = -\rho \imath \int \int A(\kappa_2) \cdot \frac{\kappa_1 \kappa_1 \kappa_1}{\kappa_1 \cdot \kappa_1} \cdot A(\kappa_1 - \kappa_2) e^{\imath \kappa_1 r} |d\kappa_1| |d\kappa_2| \quad (136)$$

Substitution in the equation of motion leads to the integro-differential equation

$$\frac{\partial}{\partial t} A(\kappa_1) = -\imath U i \cdot \kappa_1 A(\kappa_1) - \imath \int A(\kappa_2) \cdot \left(1 - \frac{\kappa_1 \kappa_1}{\kappa_1 \cdot \kappa_1}\right) \kappa_1 \cdot A(\kappa_1 - \kappa_2) |d\kappa_2| - \frac{\mu}{\rho} \kappa_1 \cdot \kappa_1 A(\kappa_1) \quad (137)$$

which expresses the evolution of the spectral representation of the velocity in an unbounded fluid

The rate of change of amplitude is expressed as the sum of three terms. The first term replaces x in the Fourier transform with $x - Ut$. The second term expresses the influence of amplitude at other points in wave number space. The third term gives the rate of viscous dissipation.

The rate of change of $A(\kappa_1)$ is influenced by the presence of $A(\kappa_2)$ in the vicinity of κ_1 . In the integrand the postmultiplication of $A(\kappa_2)$ by a tensor eliminates any component of $A(\kappa_2)$ in the direction of κ_1 . Other terms outside the integral are the products of scalars and $A(\kappa_1)$. Thus the integro-differential equation preserves the orthogonality of κ_1 and $A(\kappa_1)$.

There is no contribution to the integration where κ_2 is orthogonal to κ_1 and $A(\kappa_2)$ is parallel to κ_1 . There is no contribution to the integration where κ_2 is collinear with κ_1 and $A(\kappa_1 - \kappa_2)$ is orthogonal to κ_1 . If κ_2 is on the perpendicular bisector of κ_1 , then interchange of κ_2 and $\kappa_1 - \kappa_2$ leaves amplitudes the same but reverses the sign. There is no contribution to the integration by integration along the perpendicular bisector of κ_1 . There is a maximum contribution to the integration when κ_2 is on the perpendicular to κ_1 through the tip of κ_1 . The influence of $A(\kappa_2)$ is a pattern which is crossed by nodal lines.

DIFFUSION

During unbounded evolution the velocity deviates gradually from the boundary conditions at the surface of a plate. Velocity is injected gradually into the stream to maintain the boundary conditions. It is only at the surface of the plate that velocity is injected. Everywhere else the evolution of velocity is free. Let the plane with plate be divided into an equally spaced grid. Over each grid point there is a sine quotient function. Each sine quotient function is unity at its own grid point and is zero on every grid line which does not pass through the grid point. An analytic function can be expressed as a series in sine quotient functions. The coefficients of the terms in the series are just the values of the function at the grid points. The terms for defect in velocity are finite only for grid points within the area of the plate. The sine quotient function at any grid point can be expressed by a Fourier transform. The amplitude of the Fourier transform is a rectangle of constant density in wave number space.

Let x, y, z be Cartesian coordinates relative to a grid point with x positive downstream, with y positive across the stream, and with z positive perpendicular to the stream. Let $Q(\alpha, \beta)$ be the Fourier amplitude for the sine quotient function for the grid point and let Δ be the defect of velocity at the grid point. The velocity is injected into the laminar sublayer where the absence of convection reduces the equation of motion to

the diffusion equation

$$\frac{\partial v}{\partial t} = \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \quad (138)$$

The defect in boundary condition is concentrated initially against the plate surface and then diffuses outward. The velocity for a defect Δ is given by the equation

$$v = \Delta \frac{2}{\sqrt{\pi}} \int_{\frac{|z|}{\sqrt{4\nu t}}}^{\infty} e^{-u^2} du \int \int Q(\alpha, \beta) e^{-\nu(\alpha^2 + \beta^2)t + i(\alpha x + \beta y)} d\alpha d\beta \quad (139)$$

in which the velocity is proportional to the complementary error function. That this is a solution of the diffusion equation can be verified by direct substitution if $z \neq 0$.

The error function is symmetric with respect to the plane $z = 0$. The Fourier amplitude of the error function is the sum of an integration in the range $-\infty < z < 0$ and an integration in the range $0 < z < +\infty$. The integrals in the two ranges are complex conjugates. Integration by parts in each range and cancellation of complex conjugates lead to the equation

$$\int_{-\infty}^{+\infty} e^{-i\gamma z} \frac{2}{\sqrt{\pi}} \int_{\frac{|z|}{\sqrt{4\nu t}}}^{\infty} e^{-u^2} du dz = \frac{e^{-\gamma^2 \nu t}}{i\gamma} \frac{2}{\sqrt{\pi}} \int_{-i\gamma\sqrt{\nu t}}^{+i\gamma\sqrt{\nu t}} e^{-u^2} du \quad (140)$$

The amplitude of the error function is an even real function of γ . The velocity is given by the equation

$$v = \frac{\Delta}{2\pi} \int \int \int \frac{2}{\sqrt{\pi}} \int_{-i\gamma\sqrt{\nu t}}^{+i\gamma\sqrt{\nu t}} e^{-u^2} du \frac{Q(\alpha, \beta)}{i\gamma} e^{-\nu(\alpha^2 + \beta^2 + \gamma^2)t + i(\alpha x + \beta y + \gamma z)} d\alpha d\beta d\gamma \quad (141)$$

To verify this equation the error function can be replaced by its absolutely convergent power series, then term-by-term integration leads to the series for the arcsine of unity. The equation gives the initial velocity for the defect Δ . The velocity continues to evolve thereafter in accordance with the full equation of motion for unbounded flow.

DISCUSSION

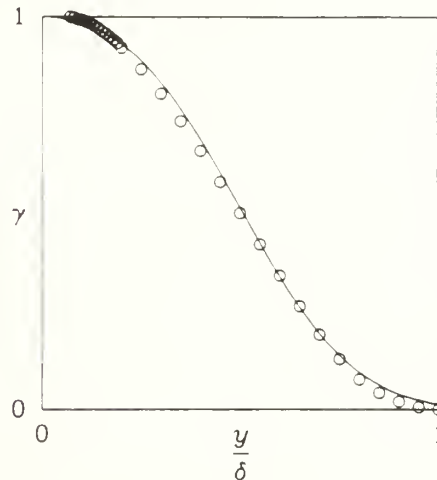
An analysis of the complete Navier-Stokes equation for Reynolds numbers less than 1218 has been given by Schwiderski and Lugt³. They found overshoot in the velocity profile and the overshoot increased with Reynolds number. There does not seem to be any confirmation for the overshoot. There is no overshoot in the Blasius profile. Validation of the Blasius theory for Reynolds numbers more than 100000 is given by the excellent agreement between measurement and theory in Figure 7.9 in the text by Schlichting⁷. It would be nice to have more experimental data for a smaller Reynolds number.

The flux of momentum across any cross section is equal to the drag upstream of the cross section. The flux of momentum is given by the integral

$$\int_0^{\infty} u(U - u) dy \quad (142)$$

where u is the instantaneous velocity in the fluid. The instantaneous velocity can be replaced by mean velocity only in the case of laminar flow. Otherwise the mean square of the instantaneous velocity is greater than the square of the mean velocity by the mean square of the fluctuation in velocity.

Heretofore it has been assumed that the blending function for a turbulent boundary layer is symmetric with respect to the middle of the boundary layer, and is zero outside the thickness of the boundary layer. It is more likely that the blending function diminishes gradually with distance. An empirical formulation is compared with experimental data in the following figure



where the curve represents Equation (35) and the circles are from a tabulation in the report by Thompson⁵.

The assumption that the horizontal component of velocity is constant everywhere in the free stream leads to a vertical component which persists to infinite distance from the boundary. However, the distribution of vertical velocity initiates a jet and diffusion of vorticity disperses the jet in the free stream.

At the leading edge of a boundary layer the flow is laminar, but the flow becomes unstable where the Reynolds number is 520, and the flow becomes fully turbulent where the Reynolds number is 500000. For a free-stream flow at 20 knots the laminar flow is unstable at only one twentieth of a millimeter downstream, and the flow is fully turbulent at five centimeters downstream. Any laminar flow in the boundary layer of a full-scale ship is insignificant.

CONCLUSION

It is concluded that the most practical representation of the mean velocity is a generalization of the empirical formulation by Schlichting

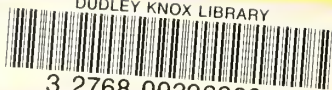
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